INDIAN STATISTICAL INSTITUTE

Probability Theory I: B. Math (Hons.) I

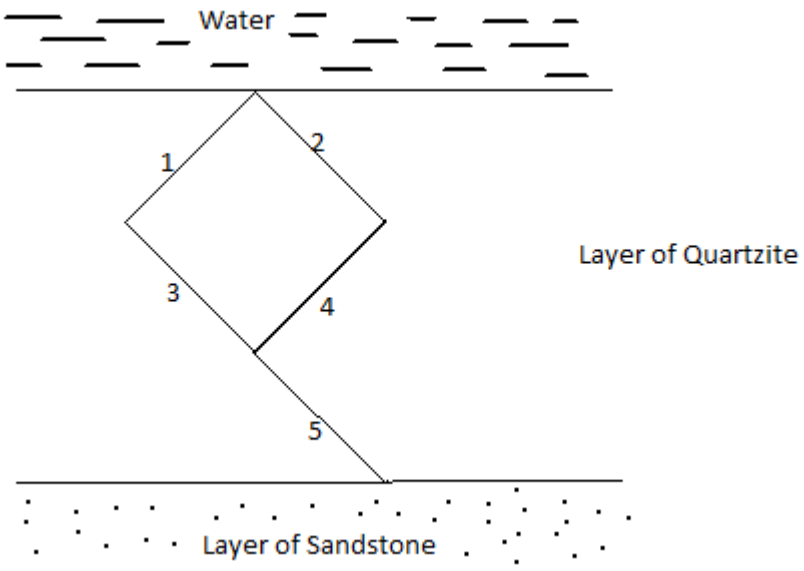
Semester I, Academic Year 2017-18

Final Exam

Date: 15/11/2017 **Total Marks: 50** **Duration: 10 am - 1 pm**

- **Show all your works and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. Consider the following schematic diagram of a drainage network model (as described in class), where each of the five paths is open with probability $p \in (0, 1)$ and the paths behave independently of each other.



(a) (7 marks) Let X be the number of open paths and Y be the indicator that water can pass through the layer of quartzite to the layer of sandstone. Find the joint probability mass function of X and Y.

(b) (3 marks) If it is given that water has passed through the layer of quartzite, find the probability that exactly 4 paths were open.

2. (3+7 = 10 marks) Let N be the number of empty poles when r flags of different colours are displayed at random on n poles arranged in a row (here $r, n \in \mathbb{N}$). Assuming that there is no limitation on the number of flags on each pole, compute the expectation and variance of N.

3. (10 marks) Students of B. Math First Year (B-I) decides to play a two-stage probabilistic cricket series with the students of B. Math Second Year (B-II) as follows. In the first stage, there will be 10 matches between them and if B-I wins n of them (clearly, $0 \leq n \leq 10$), then there will be n more matches in the second stage. Let W denote the number of matches won by B-I in the second stage. Assuming that B-I always has a 50% chance of winning a game against B-II and the results of all the games are independent of each other, calculate the probability mass function of W.

4. (10 marks) Suppose F is the cumulative distribution function of a random variable (not necessarily discrete or absolutely continuous). Show, with full justification, that

$$\lim_{x \to -\infty} F(x) = 0.$$

5. Suppose $X \sim N(0, 1)$ and $Y := X^4$.
   (a) (5 marks) Find the probability density function of Y.
   (b) (1+4 = 5 marks) Show that Y has finite second moment and compute its value.